SLIM EXCEPTIONAL SETS FOR SUMS OF CUBES

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Abstract. We investigate exceptional sets associated with various additive problems involving sums of cubes. By developing a method wherein an exponential sum over the set of exceptions is employed explicitly within the Hardy-Littlewood method, we are better able to exploit excess variables. By way of illustration, we show that the number of odd integers not divisible by 9, and not exceeding X, that fail to have a representation as the sum of 7 cubes of prime numbers, is $O(X^{23/36+\varepsilon})$. For sums of eight cubes of prime numbers, the corresponding number of exceptional integers is $O(X^{11/36+\varepsilon})$.

1. Introduction. Oftentimes in the additive theory of numbers, one encounters situations in which current technology lacks the power to establish that all large integers are represented in some prescribed manner, yet it can be shown that *almost all* positive integers are thus represented. For example, while it remains only a conjecture that all large integers are represented as the sum of four positive integral cubes, a celebrated theorem of Davenport [7] shows that almost all positive integers are represented in such a manner. To be precise, Davenport establishes that for each positive number ε, at most $O(X^{29/30+\varepsilon})$ of the natural numbers not exceeding X fail to admit a representation as the sum of four positive integral cubes. We remark that estimates for the size of this exceptional set have since been sharpened, to the extent that the exponent 29/30 may now be replaced by a number slightly smaller than 37/42 (see Brüdern [3] and Wooley [24]). One might imagine that for problems involving five or six cubes, the availability of additional variables would permit substantially sharper estimates to be obtained for the corresponding exceptional sets. Much to the chagrin of workers in the area, however, such excess variables lead only to rather modest improvements. Roughly speaking, the quality of available bounds in such problems depends on a mean value estimate over minor arcs, and traditional methods exploit excess variables via comparatively weak bounds of Weyl-type for associated exponential sums. This difficulty is universal in problems involving exceptional sets, and is particularly acute when the representations under consideration generate exponential sums for which available estimates of Weyl-type are barely non-trivial, as is the case for powers of prime numbers, for example.

In previous work devoted to sums of four squares (see Wooley [25]), we described a novel approach to certain exceptional set problems in which excess variables are more effectively exploited, and we announced our intention of providing a more comprehensive discussion of applications accessible to the underlying ideas. Our purpose in this paper is to fulfill the latter commitment with an investigation of exceptional sets stemming from problems involving sums of cubes. It transpires that such problems already generate a full menu sufficient to satisfy the enthusiast's appetite, and so we again defer to a future occasion the discussion of such issues for higher powers.

We begin our investigation of exceptional sets involving sums of cubes with a discussion of the Waring-Goldbach problem, which perhaps most clearly illustrates the advantages of our new approach over traditional treatments. We first require some notation. When s is an integer with $s \geqslant 5$, define the subsets $\mathcal{N}_s$ of $\mathbb{N}$ by

$$\begin{aligned}
\mathcal{N}_5 &= \{n \in \mathbb{N} : n \equiv 1 \pmod 2,\ n \not\equiv 0, \pm 2 \pmod 9,\ n \not\equiv 0 \pmod 7\},\\
\mathcal{N}_6 &= \{n \in \mathbb{N} : n \equiv 0 \pmod 2,\ n \not\equiv \pm 1 \pmod 9\},\\
\mathcal{N}_7 &= \{n \in \mathbb{N} : n \equiv 1 \pmod 2,\ n \not\equiv 0 \pmod 9\},\\
\mathcal{N}_s &= \{n \in \mathbb{N} : n \equiv s \pmod 2\} \quad (s \geqslant 8).
\end{aligned}$$

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It is conjectured that whenever n is a large integer with $n \in \mathcal{N}_s$, then n is represented as the sum of s cubes of primes, the implicit congruence conditions arising naturally from the observation that when p is a prime number exceeding 7, one has

$$p^3 \equiv 1 \pmod 2, \quad p^3 \equiv \pm 1 \pmod 9 \quad \text{and} \quad p^3 \equiv \pm 1 \pmod 7.$$

Motivated by this conjecture, when $s \geqslant 5$ we define $E_s(X)$ to be the number of integers $n \in \mathcal{N}_s$ not exceeding X that cannot be written as the sum of s cubes of prime numbers. Following the preparation of some technical estimates in §§2 and 3, we deploy our novel machinery in §§4–7 to obtain the bounds on $E_s(X)$ recorded in the following theorem.

Theorem 1.1. *Suppose that X is a large real number and suppose also that $5 \leqslant s \leqslant 8$. Then for every positive number ε, one has $E_s(X) \ll X^{\alpha_s+\varepsilon}$, where*

$$\alpha_5 = 35/36, \quad \alpha_6 = 17/18, \quad \alpha_7 = 23/36 \quad \text{and} \quad \alpha_8 = 11/36.$$

We remark that classical methods originating in work of Hua [17] show that $E_s(X) \ll_A X(\log X)^{-A}$ $(5 \leqslant s \leqslant 8)$, for any positive number A, and moreover that $E_s(X) \ll 1$ for $s \geqslant 9$. Very recently, Xiumen Ren [20] has sharpened Hua's estimate for $E_5(X)$ to obtain a conclusion resembling Theorem 1.1, save that the exponent α_5 is replaced by 152/153, and indeed Ren's methods are readily modified to yield bounds for $E_s(X)$ also when $s > 5$, with the exponents α_s replaced by $1 - (s-4)/153$. The estimate for $E_5(X)$ presented in Theorem 1.1 owes its superiority to a more effective analysis of the major arcs underlying the implicit application of the Hardy-Littlewood method. The bounds for $E_s(X)$ $(6 \leqslant s \leqslant 8)$ recorded in Theorem 1.1, on the other hand, owe their strength to the new ideas advertised in our opening paragraph.

As a second illustration of our methods we consider sums of cubes of smooth numbers. Here we again encounter a situation in which available estimates for the underlying exponential sums are currently rather weak. When n is a natural number, let $P(n)$ denote the largest prime divisor of n. When $s \geqslant 4$, we define $\mathcal{E}_s(X;\beta)$ to be the number of integers m with $1 \leqslant m \leqslant X$ for which the equation

$$n_1^3 + n_2^3 + \cdots + n_s^3 = m$$

fails to possess a solution $\mathbf{n} \in \mathbb{N}^s$ with $P(n_1 n_2 \dots n_s) < m^\beta$. In §8 we sketch the proof of the following theorem.

Theorem 1.2. *Suppose that X is a large real number, and that η is a sufficiently small positive number. Then for every positive number ε, one has*

$$\mathcal{E}_s(X;\eta) \ll X^{\gamma_s+\varepsilon},$$

where

$$\gamma_4 = 1 - \tfrac{1}{3}\eta, \quad \gamma_5 = 1 - \tfrac{2}{3}\eta, \quad \gamma_6 = \tfrac{2}{3} - \tfrac{1}{3}\eta \quad \text{and} \quad \gamma_7 = \tfrac{1}{3} - \tfrac{1}{3}\eta.$$

Here, the implicit constants in Vinogradov's notation may depend on ε and η.

The above estimate for $\mathcal{E}_4(X;\eta)$ is established, in essence, in the work of Brüdern and Wooley [6]. We record this bound only to provide a basis for comparison with the bounds for $\mathcal{E}_s(X;\eta)$ $(5 \leqslant s \leqslant 7)$ available from the methods of this paper. We note also that Harcos [10] has shown that $\mathcal{E}_s(X;\eta) \ll 1$ for $s \geqslant 9$, and that Brüdern and Wooley [6] have demonstrated that $\mathcal{E}_8(X;\eta) \ll 1$.

We turn our attention next to problems in which the cubes underlying the representation under consideration are not restricted to exotic sets. Denote by $R_s(n)$ the number of representations of n as the sum of s cubes of positive integers. A heuristic application of the circle method suggests that for $s \geqslant 4$, one should have the asymptotic formula

$$R_s(n) = \frac{\Gamma(4/3)^s}{\Gamma(s/3)} \mathfrak{S}_s(n) n^{s/3-1} + o(n^{s/3-1}), \tag{1.1}$$

where

$$\mathfrak{S}_s(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \Big(q^{-1} \sum_{r=1}^{q} e(ar^3/q) \Big)^s e(-an/q), \tag{1.2}$$

and $e(z)$ denotes $\exp(2\pi iz)$. It is worth noting that when $s \geqslant 4$, the relation (1.1) does indeed constitute an asymptotic formula, for it is known that the singular series $\mathfrak{S}_s(n)$ satisfies the lower bound $\mathfrak{S}_s(n) \gg 1$ whenever $s \geqslant 4$ (see Theorem 4.5 of Vaughan [22]). When $\psi(t)$ is a function of a positive variable t, denote by $\widetilde{E}_s(N;\psi)$ the number of integers n with $1 \leqslant n \leqslant N$ for which

$$\Big| R_s(n) - \frac{\Gamma(4/3)^s}{\Gamma(s/3)}\mathfrak{S}_s(n)n^{s/3-1}\Big| > n^{s/3-1}\psi(n)^{-1}. \tag{1.3}$$

In §9 we obtain the upper bound for $\widetilde{E}_7(N;\psi)$ contained in the following theorem.

Theorem 1.3. *Suppose that $\psi(t)$ is a function of a positive variable t, increasing monotonically to infinity, and satisfying the condition that when t is large, one has $\psi(t) = O((\log t)^{1-\delta})$ for some positive number δ. Then for each positive number ε, one has*

$$\widetilde{E}_7(N;\psi) \ll N^{4/9+\varepsilon}.$$

Here we note that the constant in Vinogradov's notation depends on both ε and ψ.

For comparison, the estimate $\widetilde{E}_7(N;\psi) \ll N^{1/2}$ is easily derived from work of Vaughan [21] via conventional methods. Indeed, as was essentially noted in Brüdern, Kawada and Wooley [5], it is simple to establish from the refined estimates of Boklan [2] that whenever $\varepsilon > 0$, one has

$$\widetilde{E}_{4+t}(N;\psi) \ll N^{1-t/6}(\log N)^{\varepsilon-3+t/2}\psi(t)^2 \quad (0 \leqslant t \leqslant 3).$$

It is rather disappointing that in the current state of knowledge, the methods underlying our proof of Theorem 1.3 fail to establish significant estimates for $\widetilde{E}_s(N;\psi)$ when $s < 7$. The failure of such methods for $s = 6$ can be traced to the relatively weak estimates currently available in an auxiliary representation problem involving four cubes. Denote by $R(k;P)$ the number of solutions of the diophantine equation

$$x_1^3 + x_2^3 - x_3^3 - x_4^3 = k,$$

with $1 \leqslant x_i \leqslant P$ $(1 \leqslant i \leqslant 4)$. The bound $R(k;P) \ll P^{2+\varepsilon}$ follows directly from an elementary argument involving the divisor function, and the best available estimate $R(k;P) \ll |k|^\varepsilon P^{11/6+\varepsilon}$ follows from an argument of Hooley [15] (see the proof of Lemma 2.1 of Parsell [19] for a sketch of the necessary adjustments to Hooley's argument). A formal application of the Hardy-Littlewood method, meanwhile, suggests that $R(k;P) \ll |k|^\varepsilon P^{1+\varepsilon}$. Although the former estimates are too weak to provide interesting information concerning sums of six cubes, sharper estimates can be exploited effectively through our methods. Rather than concentrate on the validity of the expected asymptotic formula, we seek instead to provide insight into the existence of a representation. Let $\widehat{E}(X)$ denote the number of natural numbers not exceeding X that are not the sum of six cubes of positive integers. We consider the consequences of the truth of a hypothesis $\mathcal{R}(A)$, which we define to be the following assertion.

Hypothesis $\mathcal{R}(A)$. *For each positive number ε, one has $R(k;P) \ll |k|^\varepsilon P^{A+\varepsilon}$.*

Thus the aforementioned conclusion stemming from Hooley's methods shows that the hypothesis $\mathcal{R}(11/6)$ is true, and it is conjectured that the hypothesis $\mathcal{R}(1)$ holds. In §10 we establish the following conditional estimate for $\widehat{E}(X)$.

Theorem 1.4. *Suppose that, for some positive number ξ with $\xi < 19/14$, the hypothesis $\mathcal{R}(\xi)$ holds. Then one has*

$$\widehat{E}(X) \ll X^{3/14}(\log X)^2.$$

The sharpest unconditional bound available for $\widehat{E}(X)$ seems to be the upper bound $\widehat{E}(X) \ll X^{23/42}$ reported in equation (1.3) of Brüdern, Kawada and Wooley [4]. We remark that by properly exploiting the mean value estimates of Wooley [23] within the methods of Brüdern, Kawada and Wooley [4], and thence within the argument of §10 below, it would be possible to replace the assumption of hypothesis $\mathcal{R}(\xi)$ above with $\mathcal{R}(19/14+\tau)$, for a suitably small positive number τ. Also, one could likewise replace the conclusion of Theorem 1.4 by the estimate $\widehat{E}(X) \ll X^{3/14-\sigma}$, for a sufficiently small positive number σ. Subject to the truth of an unproved hypothesis concerning certain Hasse-Weil L-functions, one has sharp estimates for the sixth moment of the cubic Weyl sum due to Hooley [16] and Heath-Brown [13]. Although we have not properly examined the technical details associated with the problem of inserting such an estimate into our analysis, it is clear that the limit of our methods would yield the estimate $\widehat{E}(X) \ll X^{13/42+\varepsilon}$, subject to the hypothesis of Hooley and Heath-Brown, and $\widehat{E}(X) \ll X^{1/7+\varepsilon}$, if one assumes also the hypothesis $\mathcal{R}(11/7-\delta)$, for a positive number δ.

Throughout, the letter ε will denote a sufficiently small positive number. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, implicit constants depending at most on ε, unless otherwise indicated. In an effort to simplify our analysis, we adopt the convention that whenever ε appears in a statement, then we are implicitly asserting that for each $\varepsilon > 0$ the statement holds for sufficiently large values of the main parameter. Note that the "value" of ε may consequently change from statement to statement, and hence also the dependence of implicit constants on ε. Finally, we remark that the letter p will always be reserved to denote a prime number.

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2. Auxiliary estimates. Before we are able to establish Theorem 1.1, we require some auxiliary estimates for exponential sums over cubes of prime numbers. Although the technology underlying these estimates is by now well understood, it would seem that the estimate that we require for the application at hand is not immediately available from the literature. We extract the desired estimate as concisely as we dare from work of Kawada and Wooley [18] (see also Harman [11] and Baker and Harman [1] for related estimates). It is convenient first to record some notation.

We define the multiplicative function $w(q)$ by taking

$$w(p^{3u+v}) = \begin{cases} 3p^{-u-1/2}, & \text{when } u \geqslant 0 \text{ and } v=1,\\ p^{-u-1}, & \text{when } u \geqslant 0 \text{ and } v=2,3.\end{cases}$$

Write $\omega(q)$ for the number of distinct prime divisors of q. Then it follows from the definition of $w(q)$ that for each natural number q, one has

$$w(q) \leqslant 3^{\omega(q)} q^{-1/3} \ll q^{\varepsilon-1/3},$$

for any positive number ε. Finally, we write $\tau(m)$ for the divisor function. We begin with an estimate for certain bilinear sums.

Lemma 2.1. *Let P, P', M, M', U and U' be positive real numbers with*

$$P^{1/2} \leqslant M \leqslant P, \quad P \leqslant P' \leqslant 2P \quad \text{and} \quad M \leqslant M' \leqslant 2M.$$

Suppose that (a_m) and (b_n) are sequences of complex numbers satisfying the inequalities

$$|a_m| \leqslant \tau(m) + \log m \quad \text{and} \quad |b_n| \leqslant \log n$$

for each m and n. Suppose further that α is a real number, and that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with

$$(a,q)=1, \quad 1 \leqslant q \leqslant P^{3/2} \quad \text{and} \quad |q\alpha - a| \leqslant P^{-3/2}. \tag{2.1}$$

Then one has

$$\sum_{M<m\leqslant M'} a_m \sum_{\substack{P/m<n\leqslant P'/m\\ U<n\leqslant U'}} b_n e((mn)^3\alpha) \ll PM^{\varepsilon-1/12} + (PM)^{1/2+\varepsilon} + \frac{q^\varepsilon w(q)^{1/2} P(\log P)^4}{(1+P^3|\alpha-a/q|)^{1/2}}.$$

Proof. We imitate the proof of Lemma 3.1 of Kawada and Wooley [18]. Write $N = P/(2M)$ and

$$I(n_1,n_2) = (M,2M] \cap (P/\min\{n_1,n_2\}, P'/\max\{n_1,n_2\}].$$

Then on following the argument leading to equation (3.3) of [18], we find that

$$\Big| \sum_{M<m\leqslant M'} a_m \sum_{\substack{P/m<n\leqslant P'/m \\ U<n\leqslant U'}} b_n e((mn)^3\alpha)\Big|^2 \ll M(\log P)^5(P+S_1), \tag{2.2}$$

where

$$S_1 = \sum_{N<n_1<n_2\leqslant 4N} \Big| \sum_{m\in I(n_1,n_2)} e((n_2^3-n_1^3)m^3\alpha)\Big|.$$

Denote by $\mathcal{N}$ the set of ordered pairs (n_1,n_2), with $N<n_1<n_2\leqslant 4N$, for which there exist $b\in\mathbb{Z}$ and $r\in\mathbb{N}$ with

$$(b,r)=1, \quad 1\leqslant r\leqslant 2^{-6}M^{1/2} \quad \text{and} \quad |r(n_2^3-n_1^3)\alpha-b|\leqslant \tfrac{1}{2}M^{-5/2}. \tag{2.3}$$

Then by Lemma 2.1 of [18], just as in the proof of Lemma 3.1 of that paper, we obtain

$$S_1 \ll S_2 + N^2M^{3/4+\varepsilon}, \tag{2.4}$$

where

$$S_2 = \sum_{(n_1,n_2)\in\mathcal{N}} \frac{w(r)M}{1+M^3|(n_2^3-n_1^3)\alpha-b/r|},$$

and here, the integers b and r are those defined in (2.3).

When $(n_1,n_2)\in\mathcal{N}$, we put

$$n_0=(n_1,n_2), \quad n=n_1/n_0 \quad \text{and} \quad l=(n_2-n_1)/n_0.$$

Write also $D=((n+l)^3-n^3)/l$. For each pair (n_0,l) with $1\leqslant n_0\leqslant 4N$ and $1\leqslant l\leqslant 4N/n_0$, we apply Dirichlet's approximation theorem to deduce the existence of $c\in\mathbb{Z}$ and $s\in\mathbb{N}$ with

$$(c,s)=1, \quad 1\leqslant s\leqslant M^{5/2} \quad \text{and} \quad |sn_0^3l\alpha-c|\leqslant M^{-5/2}.$$

Next write

$$T(n_0,l)=\frac{s^\varepsilon w(s)Pn_0^{-1}}{1+(P/n_0)^2M|n_0^3l\alpha-c/s|}.$$

Then on following the argument of the proof of Lemma 3.1 of [18] as far as the inequality (3.12) of that paper, we find that

$$S_2 \ll \sum_{1\leqslant n_0\leqslant 4N}\ \sum_{1\leqslant l\leqslant 4N/n_0} T(n_0,l)+P^{1+\varepsilon}. \tag{2.5}$$

Moreover, as in the argument leading to (3.13) of [18], one has $T(n_0,l)\ll PM^{\varepsilon-1/6}$, except possibly when

$$1\leqslant s\leqslant M^{1/2} \quad \text{and} \quad |sn_0^3l\alpha-c|\leqslant \tfrac{1}{2}M^{-1/2}P^{-2}. \tag{2.6}$$

For each integer n_0 satisfying $1\leqslant n_0\leqslant 4N$, we denote by $\mathcal{L}$ the set of natural numbers l with $1\leqslant l\leqslant 4N/n_0$ for which the conditions (2.6) are met. Then on writing

$$S_3=\sum_{1\leqslant n_0\leqslant 4N}\sum_{l\in\mathcal{L}} T(n_0,l), \tag{2.7}$$

we deduce from (2.4) and (2.5) that

$$S_1 \ll S_3+P^{1+\varepsilon}+P^2M^{\varepsilon-7/6}. \tag{2.8}$$

For each integer n_0 satisfying $1\leqslant n_0\leqslant 4N$, it follows from Dirichlet's approximation theorem that there exist $d\in\mathbb{Z}$ and $t\in\mathbb{N}$ with

$$(d,t)=1, \quad 1\leqslant t\leqslant P^2M^{1/2} \quad \text{and} \quad |tn_0^3\alpha-d|\leqslant M^{-1/2}P^{-2}. \tag{2.9}$$

Write

$$T_1(n_0)=\frac{t^\varepsilon w(t)Pn_0^{-2}N\log P}{1+(P/n_0)^3|n_0^3\alpha-d/t|}.$$

Then on following the argument of the proof of Lemma 3.1 of [18] as far as (3.16) and (3.17) of that paper, we deduce that

$$\sum_{l\in\mathcal{L}}T(n_0,l)\ll T_1(n_0).$$

Consequently, as in the argument leading to (3.19) of [18], one finds that

$$S_1\ll\sum_{n_0\in\mathcal{N}_0}T_1(n_0)+P^{1+\varepsilon}+P^{2+\varepsilon}M^{-7/6},\tag{2.10}$$

where $\mathcal{N}_0$ denotes the set of natural numbers n_0, with $1\leqslant n_0\leqslant M^{1/6}$, such that the integers t and d defined in (2.9) satisfy

$$1\leqslant t\leqslant M^{1/2}\quad\text{and}\quad |tn_0^3\alpha-d|\leqslant M^{1/2}P^{-3}.\tag{2.11}$$

Suppose now that $n_0\in\mathcal{N}_0$. When $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ satisfy (2.1), it follows from (2.11) that

$$\begin{aligned}|n_0^3ta-dq|&\leqslant n_0^3tP^{-3/2}+qM^{1/2}P^{-3}\\&\leqslant MP^{-3/2}+M^{1/2}P^{-3/2}<1.\end{aligned}$$

Thus we have $d/(tn_0^3)=a/q$ and $t=q/(q,n_0^3)$, and we may proceed as in the final phase of the argument of the proof of Lemma 3.1 of [18] to deduce that

$$\sum_{n_0\in\mathcal{N}_0}T_1(n_0)\ll\frac{q^\varepsilon w(q)PN(\log P)^2}{1+P^3|\alpha-a/q|}.$$

On substituting this estimate into (2.10) and recalling (2.2), the conclusion of the lemma follows immediately.

Next we consider trilinear sums, and here we again turn to the work of Kawada and Wooley [18] in order to economise on detail.

Lemma 2.2. *Let P, P', M and N be real numbers with $M\geqslant 1$, $N\geqslant 1$, $2\leqslant P\leqslant P'\leqslant 2P$,*

$$MN^3\leqslant P\quad\text{and}\quad M^3N^{-1}\leqslant P.$$

Suppose that (a_m), (b_n), (c_l) are sequences of complex numbers satisfying

$$|a_m|\leqslant 1+\log m,\quad |b_n|\leqslant 1$$

for each m and n, and with $c_l=1$ for all l, or $c_l=\log l$ for all l. Suppose further that α is a real number, and that there exist $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ satisfying (2.1). *Then one has*

$$\begin{aligned}\sum_{1\leqslant m\leqslant M}a_m\sum_{1\leqslant n\leqslant N}b_n&\sum_{P/(mn)<l\leqslant P'/(mn)}c_le((lmn)^3\alpha)\\&\ll P^{3/4+\varepsilon}(MN)^{1/4}+\frac{q^\varepsilon w(q)P(\log P)^4}{1+P^3|\alpha-a/q|}.\end{aligned}$$

Proof. This is the special case of Lemma 3.2 of [18] in which $k=3$. An inspection of the proof of the latter lemma reveals that the condition $k\geqslant 4$ imposed in its statement may be replaced with the weaker constraint $k\geqslant 3$ without impairing the conclusion.

We now arrive at the object of our endeavours within this section.

Lemma 2.3. *Let U be a real number with $U \geqslant 2$. Suppose that α is a real number, and that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying*

$$(a,q)=1, \quad 1 \leqslant q \leqslant U^{3/2} \quad \text{and} \quad |q\alpha - a| \leqslant U^{-3/2}.$$

Then one has

$$\sum_{U<p\leqslant 2U} (\log p) e(p^3\alpha) \ll U^{23/24+\varepsilon} + \frac{q^\varepsilon w(q)^{1/2} U (\log U)^5}{(1+U^3|\alpha - a/q|)^{1/2}}.$$

Proof. Equipped with Lemmata 2.1 and 2.2 above in place of Lemmata 3.1 and 3.2 of [18], one may follow the argument of the proof of Lemma 3.3 of the latter paper without serious modification to obtain the desired conclusion.

3. The major arc contribution. We are able to avoid technical discussion in our treatment of the major arc contribution relevant to our proof of Theorem 1.1, but only by employing recent work of Ren [20]. We begin by recording some notation. Let N be a sufficiently large positive number. Following the notation introduced in [20], we write

$$U=(N/20)^{1/3}, \quad P=N^{1/20} \quad \text{and} \quad Q=NP^{-1}. \tag{3.1}$$

We define the set of major arcs $\mathfrak{M}$ as the union of the intervals

$$\mathfrak{M}(q,a)=\{\alpha \in [0,1) : |q\alpha - a| \leqslant Q^{-1}\},$$

with $0 \leqslant a \leqslant q \leqslant P$ and $(a,q)=1$. We then denote the corresponding set of minor arcs by $\mathfrak{m}=[0,1)\backslash\mathfrak{M}$. Finally, we define the weighted exponential sum $g(\alpha)$ by

$$g(\alpha)=\sum_{U<p\leqslant 2U} (\log p) e(p^3\alpha).$$

Lemma 3.1. *Suppose that $5 \leqslant s \leqslant 8$, and that the integer n satisfies $N/2 < n \leqslant N$ and $n \in \mathcal{N}_s$. Then one has*

$$\int_{\mathfrak{M}} g(\alpha)^s e(-n\alpha)\,d\alpha \gg N^{s/3-1}. \tag{3.2}$$

Proof. A moment of contemplation reveals that the integral on the left hand side of (3.2) is necessarily real. With this observation in mind, we seek to show that there exist positive numbers C_s $(5 \leqslant s \leqslant 8)$ with the property that, whenever $sN/18 < n \leqslant N$ and $n \in \mathcal{N}_s$, one has

$$\int_{\mathfrak{M}} g(\alpha)^s e(-n\alpha)\,d\alpha \geqslant C_s N^{s/3-1}. \tag{3.3}$$

Furthermore, we aim to show that for any integer n one has

$$\int_{\mathfrak{M}} g(\alpha)^s e(-n\alpha)\,d\alpha \geqslant -N^{s/3-1}(\log N)^{-1}. \tag{3.4}$$

Having established the lower bound (3.3) for $5 \leqslant s \leqslant 8$, it is apparent that the conclusion of the lemma follows immediately.

We begin by discussing the desired bounds (3.3) and (3.4) when $s=5$. Write

$$S(\alpha)=\sum_{U<m\leqslant 2U} \Lambda(m) e(m^3\alpha),$$

where $\Lambda(\cdot)$ denotes the well-known von Mangoldt function. Then as a consequence of the argument employed by Ren [20] in his proof of Theorem 2 of the latter paper, one finds that there exists a positive number c_5 such that whenever $5N/18 < n \leqslant N$ and $n \in \mathcal{N}_5$, then

$$\int_{\mathfrak{M}} S(\alpha)^5 e(-n\alpha)\,d\alpha \geqslant c_5 U^2. \tag{3.5}$$

Moreover, a modest modification of this argument also reveals that for any integer n, one has

$$\int_{\mathfrak{M}} S(\alpha)^5 e(-n\alpha)\,d\alpha \geqslant -U^2(\log N)^{-1}. \tag{3.6}$$

Two comments are in order here. Firstly, our definition of U differs from that of Ren by a constant factor. A perusal of Ren's argument will, however, convince the reader that this modest adjustment is inconsequential so far as the conclusions (3.5) and (3.6) are concerned. Secondly, our definition of the set $\mathcal{N}_5$ is more restrictive than the corresponding definition implicit in the statement of Theorem 2 of [20]. It would appear that Ren made an oversight in his definition, for it is clear that neither even integers, nor those divisible by 7, can be represented as the sum of five cubes of prime numbers exceeding 7.

On noting that the measure of the arcs $\mathfrak{M}$ is $O(P^2N^{-1})$, and moreover that

$$|S(\alpha)-g(\alpha)| \leqslant \sum_{h=2}^{\infty} \sum_{U<p^h\leqslant 2U} \log p \ll U^{1/2}\log U,$$

one finds from the trivial estimate $|S(\alpha)| \ll U$ that

$$\Bigl|\int_{\mathfrak{M}} S(\alpha)^5 e(-n\alpha)\,d\alpha - \int_{\mathfrak{M}} g(\alpha)^5 e(-n\alpha)\,d\alpha\Bigr| \ll (U^{9/2}\log U)(P^2N^{-1}) \ll N^{19/30}.$$

We therefore conclude from (3.5) that there exists a positive number C_5 such that, whenever $5N/18 < n \leqslant N$ and $n \in \mathcal{N}_5$, one has

$$\int_{\mathfrak{M}} g(\alpha)^5 e(-n\alpha)\,d\alpha \geqslant c_5U^2 + O(N^{19/30}) \geqslant C_5N^{2/3}.$$

Also, for every integer n one obtains

$$\int_{\mathfrak{M}} g(\alpha)^5 e(-n\alpha)\,d\alpha \geqslant -U^2(\log N)^{-1} + O(N^{19/30}) \geqslant -N^{2/3}(\log N)^{-1}.$$

This completes the proof of the desired lower bounds (3.3) and (3.4) when $s=5$.

Suppose next that $6 \leqslant t \leqslant 8$ and that the claimed lower bounds (3.3) and (3.4) hold for $s=t-1$. Plainly, one has

$$\int_{\mathfrak{M}} g(\alpha)^t e(-n\alpha)\,d\alpha = \sum_{U<p\leqslant 2U} \log p \int_{\mathfrak{M}} g(\alpha)^s e(-(n-p^3)\alpha)\,d\alpha.$$

Moreover, one has $n-p^3 \in \mathcal{N}_s$ whenever $n \in \mathcal{N}_t$. Thus we conclude that whenever $tN/18 < n \leqslant N$ and $n \in \mathcal{N}_t$, one has

$$\int_{\mathfrak{M}} g(\alpha)^t e(-n\alpha)\,d\alpha \geqslant C_sN^{s/3-1} \sum_{\substack{U<p\leqslant 2U\\ sN/18<n-p^3\leqslant N}} \log p - N^{s/3-1}(\log N)^{-1} \sum_{U<p\leqslant 2U} \log p.$$

Then it follows from the prime number theorem that under the previous conditions on n,

$$\begin{aligned}\int_{\mathfrak{M}} g(\alpha)^t e(-n\alpha)\,d\alpha &\geqslant C_sN^{s/3-1} \sum_{U<p\leqslant (10/9)^{1/3}U} \log p - 2N^{s/3-1}U(\log N)^{-1}\\ &\geqslant (18^{-1/3}-20^{-1/3})C_sN^{t/3-1} + O(N^{t/3-1}(\log N)^{-1}).\end{aligned}$$

The desired lower bound (3.3) consequently follows with t in place of s, wherein

$$C_t = \tfrac{1}{2}(18^{-1/3}-20^{-1/3})C_{t-1}.$$

For any integer n, moreover, it follows that

$$\begin{aligned}\int_{\mathfrak{M}} g(\alpha)^t e(-n\alpha)\,d\alpha &\geqslant -N^{s/3-1}(\log N)^{-1} \sum_{U<p\leqslant 2U} \log p\\ &\geqslant -2UN^{s/3-1}(\log N)^{-1},\end{aligned}$$

and thus the lower bound (3.4) also follows with t in place of s. The claimed lower bounds (3.3) and (3.4) therefore follow for $5 \leqslant s \leqslant 8$ by induction, and this completes the proof of the lemma.

4. Five cubes of prime numbers. The preparations already behind us very nearly suffice to establish the upper bound for $E_5(X)$ recorded in Theorem 1.1. The remaining additional tools that we require cost us little further effort, and in any case prove valuable in subsequent sections. We begin with some further notation. When $5 \leqslant s \leqslant 8$, denote by $\mathcal{Z}_s(N)$ the set of integers n with $N/2 < n \leqslant N$ for which $n \in \mathcal{N}_s$, and yet the equation

$$p_1^3 + \cdots + p_s^3 = n$$

has no solution in prime numbers $p_1, \ldots, p_s$. Define the exponential sum

$$K_s(\alpha) = \sum_{n \in \mathcal{Z}_s(N)} e(n\alpha), \tag{4.1}$$

and, for the sake of convenience, write $Z_s = \text{card}(\mathcal{Z}_s(N))$. In view of the definition of $\mathcal{Z}_s(N)$, it is evident from orthogonality that

$$\int_0^1 g(\alpha)^s K_s(-\alpha) d\alpha = \sum_{n \in \mathcal{Z}_s(N)} \int_0^1 g(\alpha)^s e(-n\alpha) d\alpha = 0.$$

But by Lemma 3.1, one has

$$\begin{aligned} \int_{\mathfrak{M}} g(\alpha)^s K_s(-\alpha) d\alpha &= \sum_{n \in \mathcal{Z}_s(N)} \int_{\mathfrak{M}} g(\alpha)^s e(-n\alpha) d\alpha \\ &\gg Z_s N^{s/3-1}, \end{aligned}$$

and thus we deduce that

$$\Bigl| \int_{\mathfrak{m}} g(\alpha)^s K_s(-\alpha) d\alpha \Bigr| \gg Z_s N^{s/3-1}. \tag{4.2}$$

Our objective in this and the following three sections is to obtain an upper bound for the left hand side of (4.2), and thereby an upper bound for Z_s, when $5 \leqslant s \leqslant 8$.

Before launching our main argument in this section, we prepare the ground with some notation and an auxiliary mean value estimate. When X is a real number with $1 \leqslant X \leqslant U$, we define the set of major arcs $\mathfrak{N}(X)$ to be the union of the intervals

$$\mathfrak{N}(q, a; X) = \{\alpha \in [0,1) : |q\alpha - a| \leqslant XN^{-1}\}$$

with $0 \leqslant a \leqslant q \leqslant X$ and $(a,q) = 1$. We then define the minor arcs $\mathfrak{n}(X)$ by taking $\mathfrak{n}(X) = [0,1) \setminus \mathfrak{N}(X)$. Finally, we define the function $g^*(\alpha)$ for $\alpha \in [0,1)$ by taking

$$g^*(\alpha) = Uw(q)^{1/2}(1 + U^3|\alpha - a/q|)^{-1/2},$$

when $\alpha \in \mathfrak{N}(q, a; U) \subseteq \mathfrak{N}(U)$, and otherwise by putting $g^*(\alpha) = 0$.

Lemma 4.1. *Suppose that $1 \leqslant X \leqslant U$. Then for every positive number ε, one has*

$$\int_{\mathfrak{N}(X)} |g^*(\alpha)^4 g(\alpha)^2| d\alpha \ll U^{3+\varepsilon}.$$

Proof. The desired conclusion is an immediate consequence of Lemma 3.1 of Brüdern and Wooley [6].

Our first objective in the main part of our argument is to to derive an estimate for the 10th moment of $g(\alpha)$ restricted to the set $\mathfrak{m}$. Write $\mathfrak{P} = \mathfrak{N}(U^{1/4})$ and $\mathfrak{p} = [0,1) \setminus \mathfrak{P}$. By Dirichlet's approximation theorem, for each $\alpha \in \mathfrak{p}$, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with

$$(a,q) = 1, \quad 1 \leqslant q \leqslant U^{3/2} \quad \text{and} \quad |q\alpha - a| \leqslant U^{-3/2}.$$

But in view of the definition of $\mathfrak{p}$, whenever $q \leqslant U^{1/4}$ one has $|q\alpha - a| > U^{1/4}N^{-1}$, and thus it follows from the definition of $w(q)$ together with Lemma 2.3 that

$$\sup_{\alpha \in \mathfrak{p}} |g(\alpha)| \ll U^{23/24+\varepsilon}. \tag{4.3}$$

Consequently, an application of Hua's Lemma (see, for example, Lemma 2.5 of Vaughan [22]) yields the estimate

$$\int_{\mathfrak{p}} |g(\alpha)|^{10} d\alpha \leqslant \Bigl(\sup_{\alpha\in\mathfrak{p}} |g(\alpha)|\Bigr)^2 \int_0^1 |g(\alpha)|^8 d\alpha \ll (U^{23/24+\varepsilon})^2 U^{5+\varepsilon} \ll U^{83/12+3\varepsilon}. \tag{4.4}$$

Next write

$$I_1 = \int_{\mathfrak{m}\cap\mathfrak{P}} |g^*(\alpha)^2 g(\alpha)^8| d\alpha,$$

and observe that for every number α, it follows from Lemma 2.3 that

$$|g(\alpha)|^2 \ll (U^{23/24+\varepsilon})^2 + U^\varepsilon g^*(\alpha)^2. \tag{4.5}$$

Then on making use of the argument leading to (4.4), we find that

$$\int_{\mathfrak{m}\cap\mathfrak{P}} |g(\alpha)|^{10} d\alpha \ll U^{83/12+3\varepsilon} + U^\varepsilon I_1,$$

whence by (4.4),

$$\int_{\mathfrak{m}} |g(\alpha)|^{10} d\alpha \ll U^{83/12+3\varepsilon} + U^\varepsilon I_1. \tag{4.6}$$

However, in a manner similar to that described above, it follows from Lemma 2.3 that

$$\sup_{\alpha\in\mathfrak{m}\cap\mathfrak{P}} |g(\alpha)| \ll U^{23/24+\varepsilon} + U^{1+\varepsilon}P^{-1/6} \ll U^{1+\varepsilon}P^{-1/6}. \tag{4.7}$$

Then by combining Lemma 4.1 with Schwarz's inequality, one obtains the upper bound

$$\begin{aligned} I_1 &\ll \Bigl(\sup_{\alpha\in\mathfrak{m}\cap\mathfrak{P}} |g(\alpha)|\Bigr)^2 \Bigl(\int_{\mathfrak{P}} |g^*(\alpha)^4 g(\alpha)^2| d\alpha\Bigr)^{1/2} \Bigl(\int_{\mathfrak{m}} |g(\alpha)|^{10} d\alpha\Bigr)^{1/2} \\ &\ll (U^{1+\varepsilon}P^{-1/6})^2 (U^{3+\varepsilon})^{1/2} \Bigl(\int_{\mathfrak{m}} |g(\alpha)|^{10} d\alpha\Bigr)^{1/2}. \end{aligned}$$

On substituting this estimate into (4.6), we therefore deduce that

$$\int_{\mathfrak{m}} |g(\alpha)|^{10} d\alpha \ll U^{83/12+\varepsilon} + U^{7+\varepsilon}P^{-2/3}. \tag{4.8}$$

Finally, on applying Schwarz's inequality to (4.2) and substituting from (4.8), we obtain

$$\begin{aligned} Z_5 N^{2/3} &\ll \Bigl(\int_0^1 |K_5(\alpha)|^2 d\alpha\Bigr)^{1/2} \Bigl(\int_{\mathfrak{m}} |g(\alpha)|^{10} d\alpha\Bigr)^{1/2} \\ &\ll Z_5^{1/2} \Bigl(U^{83/12+\varepsilon} + U^{7+\varepsilon}P^{-2/3}\Bigr)^{1/2}, \end{aligned}$$

whence, on recalling (3.1), we arrive at the conclusion

$$Z_5 \ll N^{35/36+\varepsilon} + N^{1+\varepsilon}P^{-2/3} \ll N^{35/36+\varepsilon}.$$

The upper bound for $E_5(X)$ recorded in Theorem 1.1 now follows on summing over dyadic intervals.

5. Six cubes of prime numbers. Our approach to bounding $E_6(X)$ is much the same as that applied to estimate $E_5(X)$ so far as the minor arcs are concerned. However, the major arc treatment makes use of a mean value estimate useful also in the next section, and employed in spirit additionally in our discussion of Theorem 1.2.

Lemma 5.1. *For each $\varepsilon > 0$, one has*

$$\int_0^1 |g(\alpha)^4 K_s(\alpha)^2|d\alpha \ll U^\varepsilon (UZ_s^2 + U^3 Z_s). \tag{5.1}$$

Proof. By the Weyl differencing lemma (see, for example, Lemma 2.3 of Vaughan [22]), one has

$$|g(\alpha)|^4 \leqslant (2U+1) \sum_{|h_1|<U} \sum_{|h_2|<U} \sum_{x \in I} \omega(x;\mathbf{h}) e(3h_1 h_2(2x+h_1+h_2)\alpha), \tag{5.2}$$

where

$$\omega(x;\mathbf{h}) = (\log x)(\log(x+h_1))(\log(x+h_2))(\log(x+h_1+h_2)),$$

and where the inner summation is over a certain interval $I = I(h_1,h_2)$ of integers contained in $[U,2U]$, and is subject to the condition that each of the integers x, $x+h_1$, $x+h_2$ and $x+h_1+h_2$ be a prime number. By orthogonality, therefore, it follows that the integral on the left hand side of (5.1) is bounded above by the number of integral solutions of the equation

$$u_1 u_2 u_3 = n_1 - n_2, \tag{5.3}$$

with $|u_i| < 6U$ $(i = 1,2,3)$ and $n_j \in \mathcal{Z}_s(N)$ $(j = 1,2)$, and with each solution being counted with weight $3U(\log(6U))^4$.

Consider a solution $\mathbf{u}$, $\mathbf{n}$ of the equation (5.3) satisfying the associated conditions. There are plainly $O(U^2)$ choices of $\mathbf{u}$ in which one at least of the u_j is zero, and in such circumstances one necessarily has $n_1 = n_2$. Meanwhile, given any one of the $O(Z_s^2)$ possible choices for n_1 and n_2 with $n_1 \neq n_2$, a simple divisor function argument reveals that there are $O(U^\varepsilon)$ permissible choices of $\mathbf{u}$ satisfying (5.3). We therefore conclude that the total number $\mathcal{T}_1$ of solutions of (5.3) satisfies

$$\mathcal{T}_1 \ll U^2 Z_s + U^\varepsilon Z_s^2.$$

On taking account of the weights attached to each solution counted by the integral on the left hand side of (5.1), we may conclude that

$$\int_0^1 |g(\alpha)^4 K_s(\alpha)^2|d\alpha \ll U^{1+\varepsilon}(U^2 Z_s + Z_s^2).$$

The conclusion of the lemma is now immediate.

We now establish our estimate for $E_6(X)$. Defining the arcs $\mathfrak{P}$ and $\mathfrak{p}$ as in the previous section, we note initially that an argument parallel to that leading to (4.4) now yields

$$\begin{aligned} \int_{\mathfrak{p}} |g(\alpha)|^{12} d\alpha &\leqslant \Big(\sup_{\alpha \in \mathfrak{p}} |g(\alpha)|\Big)^4 \int_0^1 |g(\alpha)|^8 d\alpha \\ &\ll (U^{23/24+\varepsilon})^4 U^{5+\varepsilon} \ll U^{53/6+\varepsilon}. \end{aligned} \tag{5.4}$$

Next write

$$I_2 = \int_{\mathfrak{m}\cap\mathfrak{P}} |g(\alpha)^6 K_6(\alpha)|d\alpha.$$

Then on applying Schwarz's inequality in combination with (4.2), we deduce that

$$\begin{aligned} Z_6 N &\ll I_2 + \int_{\mathfrak{p}} |g(\alpha)^6 K_6(\alpha)|d\alpha \\ &\leqslant I_2 + \Big(\int_{\mathfrak{p}} |g(\alpha)|^{12} d\alpha\Big)^{1/2} \Big(\int_0^1 |K_6(\alpha)|^2 d\alpha\Big)^{1/2}, \end{aligned}$$

whence by (5.4),

$$Z_6 N \ll I_2 + Z_6^{1/2} U^{53/12+\varepsilon}. \tag{5.5}$$

In order to estimate I_2, we note from (4.5) that

$$I_2 \ll U^{23/12+\varepsilon} I_3 + U^\varepsilon I_4, \tag{5.6}$$

where

$$I_3 = \int_0^1 |g(\alpha)^4 K_6(\alpha)| d\alpha$$

and

$$I_4 = \int_{\mathfrak{m}\cap\mathfrak{P}} |g^*(\alpha)^2 g(\alpha)^4 K_6(\alpha)| d\alpha.$$

By Schwarz's inequality, it follows from Hua's lemma that

$$\begin{aligned} I_3 &\leqslant \Big(\int_0^1 |g(\alpha)|^8 d\alpha \Big)^{1/2} \Big(\int_0^1 |K_6(\alpha)|^2 d\alpha \Big)^{1/2} \\ &\ll U^{5/2+\varepsilon} Z_6^{1/2}. \end{aligned} \tag{5.7}$$

A second application of Schwarz's inequality, meanwhile, reveals that

$$I_4 \leqslant \Big(\sup_{\alpha\in\mathfrak{m}\cap\mathfrak{P}} |g(\alpha)| \Big) \Big(\int_{\mathfrak{P}} |g^*(\alpha)^4 g(\alpha)^2| d\alpha \Big)^{1/2} \Big(\int_0^1 |g(\alpha)^4 K_6(\alpha)^2| d\alpha \Big)^{1/2}.$$

Then it follows from (4.7) and Lemmata 4.1 and 5.1 that

$$I_4 \ll U^{1+\varepsilon} P^{-1/6} (U^{3+\varepsilon})^{1/2} (U^{1+\varepsilon} Z_6^2 + U^{3+\varepsilon} Z_6)^{1/2}. \tag{5.8}$$

On substituting (5.7) and (5.8) into (5.6), we therefore conclude from (5.5) and (3.1) that

$$Z_6 N \ll N^{1+\varepsilon} P^{-1/6} Z_6 + N^{53/36+\varepsilon} Z_6^{1/2} + N^{4/3+\varepsilon} P^{-1/6} Z_6^{1/2},$$

whence

$$Z_6 \ll N^{17/18+\varepsilon} + N^{2/3+\varepsilon} P^{-1/3} \ll N^{17/18+\varepsilon}.$$

The upper bound for $E_6(X)$ recorded in Theorem 1.1 is now immediate on summing over dyadic intervals.

6. Seven cubes of prime numbers. Our treatment of the exceptional set for sums of seven cubes of prime numbers makes use of the mean value estimate recorded in Lemma 5.1, and also requires a new mean value estimate that we record in Lemma 6.2 below. Before we establish the latter estimate, we prepare the ground with an auxiliary bound.

Lemma 6.1. *For each $\varepsilon > 0$, one has*

$$\int_0^1 |g(\alpha)^2 K_s(\alpha)^2| d\alpha \ll U^\varepsilon (U Z_s + Z_s^2). \tag{6.1}$$

Proof. The mean value on the left hand side of (6.1) counts the number of solutions of the equation

$$p_1^3 - p_2^3 = n_1 - n_2,$$

with $U < p_i \leqslant 2U$ $(i = 1, 2)$ and $n_j \in \mathcal{Z}_s(N)$ $(j = 1, 2)$, where each solution is counted with weight $(\log p_1)(\log p_2)$. There are plainly $O(Z_s U/\log U)$ solutions of this equation with $n_1 = n_2$ and $p_1^3 = p_2^3$. Given any one of the $O(Z_s^2)$ available choices of n_1 and n_2 with $n_1 \neq n_2$, meanwhile, one may apply an elementary estimate for the divisor function to show that there are $O(U^\varepsilon)$ possible choices for $p_1 - p_2$ and $p_1^2 + p_1 p_2 + p_2^2$, whence also for p_1 and p_2. We therefore conclude that

$$\int_0^1 |g(\alpha)^2 K_s(\alpha)^2| d\alpha \ll (\log U)^2 (Z_s U/\log U + U^\varepsilon Z_s^2),$$

and the conclusion of the lemma follows immediately.

We are now able to establish the new mean value estimate crucial in this section and the next.

Lemma 6.2. *For each $\varepsilon > 0$, one has*

$$\int_0^1 |g(\alpha)^6 K_s(\alpha)^2| d\alpha \ll U^\varepsilon (U^3 Z_s^2 + U^4 Z_s). \tag{6.2}$$

Proof. On recalling (5.2), it follows from orthogonality that the integral on the left hand side of (6.2) is bounded above by the number of integral solutions of the equation

$$u_1 u_2 u_3 = (p_1^3 - p_2^3) - (n_1 - n_2), \tag{6.3}$$

with $|u_i| < 6U$ $(i = 1, 2, 3)$, $U < p_j \leqslant 2U$ $(j = 1, 2)$ and $n_l \in \mathcal{Z}_s(N)$ $(l = 1, 2)$, and with each solution being counted with weight $3U(\log(6U))^6$.

Consider a solution $\mathbf{u}$, $\mathbf{p}$, $\mathbf{n}$ of the equation (6.3) satisfying the associated conditions. There are plainly $O(U^2)$ choices of $\mathbf{u}$ in which one at least of the u_j is zero, and in such circumstances one necessarily has

$$p_1^3 - p_2^3 = n_1 - n_2. \tag{6.4}$$

It follows that the number of solutions, $\mathcal{T}_2$, of this type satisfies

$$\mathcal{T}_2 \ll U^2 \int_0^1 |g(\alpha)^2 K_s(\alpha)^2| d\alpha,$$

whence by Lemma 6.1,

$$\mathcal{T}_2 \ll U^{2+\varepsilon}(U Z_s + Z_s^2). \tag{6.5}$$

On the other hand, given any of the $O(U^2 Z_s^2)$ possible choices for p_1, p_2, n_1, n_2 in which the equation (6.4) is not satisfied, a simple divisor function argument ensures that there are $O(U^\varepsilon)$ permissible choices of $\mathbf{u}$ satisfying (6.3). Thus the number of solutions, $\mathcal{T}_3$, of this complementary type satisfies

$$\mathcal{T}_3 \ll U^{2+\varepsilon} Z_s^2. \tag{6.6}$$

On combining the estimates (6.5) and (6.6) and accounting for the inherent weights, we may conclude that

$$\int_0^1 |g(\alpha)^6 K_s(\alpha)^2| d\alpha \ll U^{1+\varepsilon}(U^3 Z_s + U^2 Z_s^2).$$

This completes the proof of the lemma.

We are now equipped to establish our estimate for $E_7(X)$. We define the arcs $\mathfrak{P}$ and $\mathfrak{p}$ as in the previous sections, and begin by applying Schwarz's inequality to obtain

$$\int_{\mathfrak{p}} |g(\alpha)^7 K_7(\alpha)| d\alpha \leqslant \Big(\int_0^1 |g(\alpha)^4 K_7(\alpha)^2| d\alpha \Big)^{1/2} \Big(\int_{\mathfrak{p}} |g(\alpha)|^{10} d\alpha \Big)^{1/2}.$$

Then as a consequence of Lemma 5.1 and the inequality (4.4), we deduce that

$$\begin{aligned}\int_{\mathfrak{p}} |g(\alpha)^7 K_7(\alpha)| d\alpha &\ll U^\varepsilon (U Z_7^2 + U^3 Z_7)^{1/2} (U^{83/12})^{1/2} \\ &\ll U^{95/24+\varepsilon} (Z_7^2 + U^2 Z_7)^{1/2}.\end{aligned}$$

Next write

$$I_5 = \int_{\mathfrak{m} \cap \mathfrak{P}} |g(\alpha)^7 K_7(\alpha)| d\alpha.$$

Then we conclude from (4.2) and (3.1) that

$$\begin{aligned}Z_7 N^{4/3} &\ll I_5 + \int_{\mathfrak{p}} |g(\alpha)^7 K_7(\alpha)| d\alpha \\ &\ll I_5 + N^{4/3 - 1/72 + \varepsilon} Z_7 + N^{5/3 - 1/72 + \varepsilon} Z_7^{1/2}.\end{aligned} \tag{6.7}$$

We estimate I_5 by applying Lemma 2.3 to obtain

$$I_5 \ll U^{23/24+\varepsilon} I_6 + U^\varepsilon I_7, \tag{6.8}$$

where

$$I_6 = \int_0^1 |g(\alpha)^6 K_7(\alpha)|d\alpha$$

and

$$I_7 = \int_{\mathfrak{m}\cap\mathfrak{P}} |g^*(\alpha) g(\alpha)^6 K_7(\alpha)|d\alpha.$$

On combining Schwarz's inequality with Hua's Lemma and Lemma 5.1, we find that

$$\begin{aligned} I_6 &\leqslant \Big(\int_0^1 |g(\alpha)|^8 d\alpha \Big)^{1/2} \Big(\int_0^1 |g(\alpha)^4 K_7(\alpha)^2| d\alpha \Big)^{1/2} \\ &\ll (U^{5+\varepsilon})^{1/2} (U^{1+\varepsilon} Z_7^2 + U^{3+\varepsilon} Z_7)^{1/2}. \end{aligned} \tag{6.9}$$

A second application of Schwarz's inequality, moreover, reveals that

$$\begin{aligned} I_7 \leqslant &\Big(\sup_{\alpha\in\mathfrak{m}\cap\mathfrak{P}} |g(\alpha)| \Big)^{1/2} \Big(\int_{\mathfrak{P}} |g^*(\alpha)^4 g(\alpha)^2| d\alpha \Big)^{1/4} \\ &\times \Big(\int_0^1 |g(\alpha)^6 K_7(\alpha)^2| d\alpha \Big)^{1/2} \Big(\int_0^1 |g(\alpha)|^8 d\alpha \Big)^{1/4}. \end{aligned}$$

Then by Lemmata 4.1 and 6.2, Hua's lemma and (4.7), we deduce that

$$\begin{aligned} I_7 &\ll U^{1/2+\varepsilon} P^{-1/12} (U^3)^{1/4} (U^3 Z_7^2 + U^4 Z_7)^{1/2} (U^5)^{1/4} \\ &\ll U^{4+\varepsilon} P^{-1/12} Z_7 + U^{9/2+\varepsilon} P^{-1/12} Z_7^{1/2}. \end{aligned} \tag{6.10}$$

On substituting (6.9) and (6.10) into (6.8), we thus conclude from (3.1) and (6.7) that

$$\begin{aligned} Z_7 N^{4/3} \ll &N^{4/3+\varepsilon} P^{-1/12} Z_7 + N^{4/3-1/72+\varepsilon} Z_7 \\ &+ N^{5/3-1/72+\varepsilon} Z_7^{1/2} + N^{3/2+\varepsilon} P^{-1/12} Z_7^{1/2}, \end{aligned}$$

whence

$$Z_7 \ll N^{23/36+\varepsilon} + N^{1/3+\varepsilon} P^{-1/6} \ll N^{23/36+\varepsilon}.$$

The estimate for $E_7(X)$ recorded in Theorem 1.1 now follows on summing over dyadic intervals.

7. Eight cubes of prime numbers. No further apparatus being required for our analysis of $E_8(X)$, we launch our argument for this case immediately. First, by Schwarz's inequality, we find that

$$\int_{\mathfrak{p}} |g(\alpha)^8 K_8(\alpha)| d\alpha \leqslant \Big(\int_0^1 |g(\alpha)^6 K_8(\alpha)^2| d\alpha \Big)^{1/2} \Big(\int_{\mathfrak{p}} |g(\alpha)|^{10} d\alpha \Big)^{1/2}.$$

Then by Lemma 6.2 and the inequality (4.4), we deduce that

$$\begin{aligned} \int_{\mathfrak{p}} |g(\alpha)^8 K_8(\alpha)| d\alpha &\ll U^\varepsilon (U^3 Z_8^2 + U^4 Z_8)^{1/2} (U^{83/12})^{1/2} \\ &\ll U^{119/24+\varepsilon} (Z_8^2 + U Z_8)^{1/2}. \end{aligned}$$

Next write

$$I_8 = \int_{\mathfrak{m}\cap\mathfrak{P}} |g(\alpha)^8 K_8(\alpha)| d\alpha.$$

Then we obtain from (4.2) and (3.1) the inequality

$$\begin{aligned} Z_8 N^{5/3} &\ll I_8 + \int_{\mathfrak{p}} |g(\alpha)^8 K_8(\alpha)| d\alpha \\ &\ll I_8 + N^{5/3-1/72+\varepsilon} Z_8 + N^{11/6-1/72+\varepsilon} Z_8^{1/2}. \end{aligned} \tag{7.1}$$

We estimate I_8 by first applying Lemma 2.3 to obtain

$$I_8 \ll U^{23/24+\varepsilon} I_9 + U^{\varepsilon} I_{10}, \tag{7.2}$$

where

$$I_9 = \int_0^1 |g(\alpha)^7 K_8(\alpha)| d\alpha$$

and

$$I_{10} = \int_{\mathfrak{m}\cap\mathfrak{P}} |g^*(\alpha) g(\alpha)^7 K_8(\alpha)| d\alpha.$$

By Schwarz's inequality combined with Hua's lemma and Lemma 6.2, we find that

$$\begin{aligned} I_9 &\leqslant \Big(\int_0^1 |g(\alpha)|^8 d\alpha \Big)^{1/2} \Big(\int_0^1 |g(\alpha)^6 K_8(\alpha)^2| d\alpha \Big)^{1/2} \\ &\ll (U^{5+\varepsilon})^{1/2} (U^{3+\varepsilon} Z_8^2 + U^{4+\varepsilon} Z_8)^{1/2}. \end{aligned} \tag{7.3}$$

Next, by Hölder's inequality together with Hua's lemma, Lemma 4.1 and (4.7), we find that

$$\begin{aligned} I_{10} &\leqslant K_8(0) \Big(\sup_{\alpha\in\mathfrak{m}} |g(\alpha)| \Big)^{1/2} \Big(\int_0^1 |g(\alpha)|^8 d\alpha \Big)^{3/4} \Big(\int_{\mathfrak{P}} |g^*(\alpha)^4 g(\alpha)^2| d\alpha \Big)^{1/4} \\ &\ll Z_8 U^{1/2+\varepsilon} P^{-1/12} (U^{5+\varepsilon})^{3/4} (U^{3+\varepsilon})^{1/4} \\ &\ll Z_8 U^{5+\varepsilon} P^{-1/12}. \end{aligned} \tag{7.4}$$

Thus, on substituting (7.3) and (7.4) into (7.2), we conclude from (3.1) and (7.1) that

$$Z_8 N^{5/3} \ll N^{5/3+\varepsilon} P^{-1/12} Z_8 + N^{5/3-1/72+\varepsilon} Z_8 + N^{11/6-1/72+\varepsilon} Z_8^{1/2},$$

whence

$$Z_8 \ll N^{11/36+\varepsilon}.$$

Consequently, the estimate for $E_8(X)$ recorded in Theorem 1.1 in this case again follows instantly on summing over dyadic intervals.

8. Waring's problem for cubes of smooth numbers. The object of this section is to prove Theorem 1.2. Since this theorem is of somewhat less interest than the remaining results of this paper, we aim here to be concise, and we feel free to leave the verification of certain details to the reader. We begin by introducing some notation drawn from §5 of Brüdern and Wooley [6]. Let δ be a sufficiently small positive number. When P, Q and R are positive real numbers, we write

$$\mathcal{A}^*(Q,R) = \{ n \in [1,Q] \cap \mathbb{Z} : p \text{ prime}, p|n \Rightarrow R^{1-\delta} < p \leqslant R \},$$

and then define the set $\mathcal{C}(P,R)$ of smooth numbers by

$$\mathcal{C}(P,R) = \{ lm : m \in \mathcal{A}^*(PR^{\delta-1}, R) \text{ and } 1 \leqslant l \leqslant R^{1-\delta} \}.$$

We note here that each element of $\mathcal{C}(P,R)$ is uniquely represented in the form lm described in the definition of this set. We remark also that whenever A and B are fixed real numbers with $B > A \geqslant 1$, and P and R are large real numbers with $R^A \leqslant P \leqslant R^B$, then it follows from Friedlander [8] that

$$\frac{P}{\log R} \ll_{A,B} \text{card}(\mathcal{C}(P,R)) \ll_{A,B} \frac{P}{\log R}. \tag{8.1}$$

Define the exponential sum $h(\alpha) = h(\alpha; P, R)$ by

$$h(\alpha; P, R) = \sum_{x \in \mathcal{C}(P,R)} e(\alpha x^3),$$

and when $4 \leqslant s \leqslant 7$, write

$$T_s(n) = \int_0^1 h(\alpha)^s e(-n\alpha) d\alpha.$$

We apply the Hardy-Littlewood method to estimate $T_s(n)$ on average. Let N be large, and write $P = N^{1/3}$ and $R = (N/2)^\eta$, where we suppose that η is a sufficiently small, but fixed, positive number. We then define two Hardy-Littlewood dissections by taking

$$\mathfrak{W} = \mathfrak{N}(P^{1/3}), \quad \mathfrak{V} = \mathfrak{N}(R^{1-\delta}), \quad \mathfrak{w} = [0,1) \setminus \mathfrak{W}, \quad \mathfrak{v} = [0,1) \setminus \mathfrak{V},$$

where $\mathfrak{N}(X)$ is defined as in the preamble to Lemma 4.1 above. The following three lemmata provide the tools essential to the analysis of this section.

Lemma 8.1. *Suppose that* $4 \leqslant s \leqslant 7$ *and* $N/2 < n \leqslant N$. *Then one has*

$$\int_{\mathfrak{V}} h(\alpha)^s e(-n\alpha) d\alpha \gg N^{s/3-1} (\log N)^{-s}.$$

Proof. A simple modification of the argument of §5 of Brüdern and Wooley [6] establishes the desired conclusion in all essentials. Our use of $s \geqslant 4$ variables in this instance rather than 4 variables therein is, of course, inconsequential. Also, our modified definition of the set $\mathcal{C}(P, R)$ brings about only cosmetic alterations in the implicit proof of the desired lower bound.

Lemma 8.2. *For each* $\varepsilon > 0$, *one has*

$$\sup_{\alpha \in \mathfrak{v}} |h(\alpha)| \ll P(R^{1-\delta})^{\varepsilon - 1/6}.$$

Proof. Define the function $h^*(\alpha)$ for $\alpha \in [0, 1)$ by taking

$$h^*(\alpha) = Pw(q)^{1/2} (1 + P^3 |\alpha - a/q|)^{-1/3},$$

when $\alpha \in \mathfrak{N}(q, a; P^{1/3}) \subseteq \mathfrak{W}$, and otherwise by putting $h^*(\alpha) = 0$. Then the argument of [6] leading from equation (5.7) to (5.8) of that paper establishes that for each $\varepsilon > 0$, one has

$$h(\alpha) \ll P^\varepsilon h^*(\alpha) + P^{17/18+\varepsilon},$$

uniformly in α. Thus the estimate recorded in the lemma follows from the definition of $w(q)$ whenever η is sufficiently small.

Lemma 8.3. *For each* $\varepsilon > 0$, *one has*

$$\int_{\mathfrak{v}} |h(\alpha)|^8 d\alpha \ll P^5 (R^{1-\delta})^{\varepsilon - 1/3}.$$

Proof. This conclusion is a slightly modified version of Lemma 5.2 of [6]. We begin by observing that the argument of the latter paper leading to equation (5.8) therein readily yields the estimate

$$\int_{\mathfrak{w}} |h(\alpha)|^8 d\alpha \ll P^5 R^{-1},$$

whenever the positive number η is sufficiently small. Also, it follows from the argument leading to equation (5.9) of that paper that

$$\int_{\mathfrak{W}} |h(\alpha)|^6 d\alpha \ll P^{3+\varepsilon}.$$

Thus we deduce from Lemma 8.2 that

$$\int_{\mathfrak{v}\cap\mathfrak{W}} |h(\alpha)|^8 d\alpha \leqslant \Big(\sup_{\alpha\in\mathfrak{v}} |h(\alpha)|\Big)^2 \int_{\mathfrak{W}} |h(\alpha)|^6 d\alpha$$
$$\ll (P(R^{1-\delta})^{\varepsilon-1/6})^2 P^{3+\varepsilon}.$$

We consequently conclude that

$$\int_{\mathfrak{v}} |h(\alpha)|^8 d\alpha \leqslant \int_{\mathfrak{w}} |h(\alpha)|^8 d\alpha + \int_{\mathfrak{v}\cap\mathfrak{W}} |h(\alpha)|^8 d\alpha$$
$$\ll P^5 R^{-1} + P^{5+\varepsilon}(R^{1-\delta})^{2\varepsilon-1/3},$$

and the desired estimate follows immediately.

When $4 \leqslant s \leqslant 7$, we now denote by $\mathcal{Z}_s(N)$ the set of integers n with $N/2 < n \leqslant N$ for which the equation

$$m_1^3 + \cdots + m_s^3 = n$$

has no solution in integers m_i with $P(m_i) < n^\eta$. We define the exponential sum $K_s(\alpha)$ as in (4.1), and write $Z_s = \mathrm{card}(\mathcal{Z}_s(N))$. It is evident in the situation at hand that

$$\int_0^1 h(\alpha)^s K_s(-\alpha) d\alpha = \sum_{n\in\mathcal{Z}_s(N)} \int_0^1 h(\alpha)^s e(-n\alpha) d\alpha = 0.$$

Then Lemma 8.1 leads to the relation

$$\int_{\mathfrak{V}} h(\alpha)^s K_s(-\alpha) d\alpha = \sum_{n\in\mathcal{Z}_s(N)} \int_{\mathfrak{V}} h(\alpha)^s e(-n\alpha) d\alpha$$
$$\gg Z_s N^{s/3-1}(\log N)^{-s},$$

and so we deduce that

$$\Big|\int_{\mathfrak{v}} h(\alpha)^s K_s(-\alpha) d\alpha\Big| \gg Z_s N^{s/3-1}(\log N)^{-s}. \tag{8.2}$$

With the above prerequisites in hand, we dispose swiftly of the arguments required to establish Theorem 1.2, beginning with the estimation of $\mathcal{E}_4(X;\eta)$. This conclusion is due, in all essentials, to Brüdern and Wooley [6]. We simply observe that by Schwarz's inequality in combination with Lemma 8.3, one has

$$\Big|\int_{\mathfrak{v}} h(\alpha)^4 K_4(-\alpha) d\alpha\Big| \leqslant \Big(\int_{\mathfrak{v}} |h(\alpha)|^8 d\alpha\Big)^{1/2} \Big(\int_0^1 |K_4(\alpha)|^2 d\alpha\Big)^{1/2}$$
$$\ll P^{5/2}(R^{1-\delta})^{\varepsilon-1/6} Z_4^{1/2},$$

whence by (8.2),

$$Z_4 N^{1/3}(\log N)^{-4} \ll Z_4^{1/2} N^{5/6}(R^{1-\delta})^{\varepsilon-1/6}.$$

We may therefore conclude that

$$Z_4 \ll N^{1-\eta/3+\varepsilon},$$

provided that we choose δ sufficiently small in terms of ε. The estimate for $\mathcal{E}_4(X;\eta)$ recorded in Theorem 1.2 now follows on summing over dyadic intervals.

Next, on applying Schwarz's inequality in combination with Lemmata 8.2 and 8.3, one obtains

$$\Big|\int_{\mathfrak{v}} h(\alpha)^5 K_5(-\alpha) d\alpha\Big| \leqslant \Big(\sup_{\alpha\in\mathfrak{v}} |h(\alpha)|\Big) \Big(\int_{\mathfrak{v}} |h(\alpha)|^8 d\alpha\Big)^{1/2} \Big(\int_0^1 |K_5(\alpha)|^2 d\alpha\Big)^{1/2}$$
$$\ll P(R^{1-\delta})^{\varepsilon-1/6}(P^5(R^{1-\delta})^{\varepsilon-1/3})^{1/2} Z_5^{1/2},$$

whence by (8.2),

$$Z_5N^{2/3}(\log N)^{-5} \ll Z_5^{1/2}N^{7/6}(R^{1-\delta})^{\varepsilon-1/3}.$$

We thus obtain

$$Z_5 \ll N^{1-2\eta/3+\varepsilon},$$

and the desired estimate for $\mathcal{E}_5(X;\eta)$ follows as before.

In order to tackle the estimation of $\mathcal{E}_6(X;\eta)$, we begin by noting that the argument of the proof of Lemma 5.1 is easily modified to provide the estimate

$$\int_0^1 |h(\alpha)^4K_6(\alpha)^2|d\alpha \ll P^\varepsilon(PZ_6^2+P^3Z_6).$$

Thus, on applying Schwarz's inequality together with Lemma 8.3, we now obtain

$$\begin{aligned}\Bigl|\int_{\mathfrak{v}} h(\alpha)^6K_6(-\alpha)d\alpha\Bigr| &\leqslant \Bigl(\int_0^1 |h(\alpha)^4K_6(\alpha)^2|d\alpha\Bigr)^{1/2}\Bigl(\int_{\mathfrak{v}} |h(\alpha)|^8d\alpha\Bigr)^{1/2}\\ &\ll P^\varepsilon(PZ_6^2+P^3Z_6)^{1/2}(P^5(R^{1-\delta})^{\varepsilon-1/3})^{1/2},\end{aligned}$$

whence by (8.2),

$$Z_6N(\log N)^{-6} \ll N^{1+\varepsilon}(R^{1-\delta})^{\varepsilon-1/6}Z_6 + N^{4/3+\varepsilon}(R^{1-\delta})^{\varepsilon-1/6}Z_6^{1/2}.$$

We thus obtain

$$Z_6 \ll N^{2/3+\varepsilon}(R^{1-\delta})^{\varepsilon-1/3},$$

and the conclusion required follows as in the previous cases.

Finally, on modifying the argument of the proof of Lemma 6.2 above, one finds that

$$\int_0^1 |h(\alpha)^6K_7(\alpha)^2|d\alpha \ll P^\varepsilon(P^3Z_7^2+P^4Z_7).$$

Then applying Schwarz's inequality with Lemma 8.3, we deduce that

$$\begin{aligned}\Bigl|\int_{\mathfrak{v}} h(\alpha)^7K_7(-\alpha)d\alpha\Bigr| &\leqslant \Bigl(\int_0^1 |h(\alpha)^6K_7(\alpha)^2|d\alpha\Bigr)^{1/2}\Bigl(\int_{\mathfrak{v}} |h(\alpha)|^8d\alpha\Bigr)^{1/2}\\ &\ll P^\varepsilon(P^3Z_7^2+P^4Z_7)^{1/2}\Bigl(P^5(R^{1-\delta})^{\varepsilon-1/3}\Bigr)^{1/2},\end{aligned}$$

whence by (8.2),

$$Z_7N^{4/3}(\log N)^{-7} \ll N^{4/3+\varepsilon}(R^{1-\delta})^{\varepsilon-1/6}Z_7 + N^{3/2+\varepsilon}(R^{1-\delta})^{\varepsilon-1/6}Z_7^{1/2}.$$

We therefore conclude that

$$Z_7 \ll N^{1/3+\varepsilon}(R^{1-\delta})^{\varepsilon-1/3},$$

and we complete the proof of the claimed estimate for $\mathcal{E}_7(X;\eta)$ as in the previous cases. This completes the proof of Theorem 1.2.

9. The asymptotic formula for seven cubes. Our proof of Theorem 1.3 involves only modest prerequisites easily accomodated en passant. We therefore launch our first broadside immediately. Let N be a large positive number, and let $\psi(t)$ be a function of the type described in the statement of Theorem 1.3. We denote by $\mathcal{Z}(N)$ the set of integers n with $N/2 < n \leqslant N$ for which the inequality (1.3) holds with $s = 7$, and we abbreviate $\text{card}(\mathcal{Z}(N))$ to Z.

Write $P = [N^{1/3}]$ and define

$$f(\alpha) = \sum_{1\leqslant x\leqslant P} e(\alpha x^3).$$

By orthogonality, for each integer n with $N/2 < n \leqslant N$ one has

$$R_7(n) = \int_0^1 f(\alpha)^7 e(-n\alpha)d\alpha. \tag{9.1}$$

Let $\mathfrak{M}$ denote the union of the intervals

$$\mathfrak{M}(q,a) = \{\alpha \in [0,1) : |q\alpha - a| \leqslant \tfrac{1}{6}PN^{-1}\},$$

with $0 \leqslant a \leqslant q \leqslant \frac{1}{6}P$ and $(a,q) = 1$. Then by Theorem 4.4 of [22], there is a positive number τ such that whenever $N/2 < n \leqslant N$, one has

$$\int_{\mathfrak{M}} f(\alpha)^7 e(-n\alpha)d\alpha = \frac{\Gamma(4/3)^7}{\Gamma(7/3)}\mathfrak{S}_7(n)n^{4/3} + O(n^{4/3-\tau}). \tag{9.2}$$

Now write $\mathfrak{m} = [0,1) \setminus \mathfrak{M}$. Then for $n \in \mathcal{Z}(N)$, it follows from (1.3), (9.1) and (9.2) that

$$\Big|\int_{\mathfrak{m}} f(\alpha)^7 e(-n\alpha)d\alpha\Big| > \tfrac{1}{2}n^{4/3}\psi(n)^{-1}. \tag{9.3}$$

Next define the complex number η_n by taking $\eta_n = 0$ for $n \notin \mathcal{Z}(N)$, and when $n \in \mathcal{Z}(N)$ by means of the equation

$$\Big|\int_{\mathfrak{m}} f(\alpha)^7 e(-n\alpha)d\alpha\Big| = \eta_n \int_{\mathfrak{m}} f(\alpha)^7 e(-n\alpha)d\alpha.$$

Of course, one has $|\eta_n| = 1$ whenever η_n is non-zero. In view of (9.3), one finds that

$$\begin{aligned} n^{4/3}\psi(n)^{-1}\mathrm{card}(\mathcal{Z}(N)) &\ll \sum_{N/2<n\leqslant N} \eta_n \int_{\mathfrak{m}} f(\alpha)^7 e(-n\alpha)d\alpha \\ &= \int_{\mathfrak{m}} f(\alpha)^7 K(-\alpha)d\alpha, \end{aligned} \tag{9.4}$$

where

$$K(\alpha) = \sum_{N/2<n\leqslant N} \eta_n e(n\alpha). \tag{9.5}$$

On applying Schwarz's inequality to (9.4), one deduces that

$$N^{4/3}\psi(N)^{-1}Z \ll T_1^{1/2}T_2^{1/2}, \tag{9.6}$$

where

$$T_1 = \int_{\mathfrak{m}} |f(\alpha)|^{12}d\alpha \quad \text{and} \quad T_2 = \int_0^1 |f(\alpha)K(\alpha)|^2 d\alpha. \tag{9.7}$$

We estimate T_1 immediately by noting that, on combining the refined estimates of Hall and Tenenbaum [9] for Hooley's Δ-function with the proof of Lemma 1 of Vaughan [21], one obtains

$$\sup_{\alpha\in\mathfrak{m}} |f(\alpha)| \ll P^{3/4}(\log P)^{1/4+\varepsilon}.$$

Thus, on exploiting the estimate

$$\int_{\mathfrak{m}} |f(\alpha)|^8 d\alpha \ll P^5(\log P)^{\varepsilon-3}$$

due to Boklan [2], we obtain the upper bound

$$T_1 \leqslant \Big(\sup_{\alpha\in\mathfrak{m}} |f(\alpha)|\Big)^4 \int_{\mathfrak{m}} |f(\alpha)|^8 d\alpha \ll P^8(\log P)^{\varepsilon-2}. \tag{9.8}$$

By orthogonality, meanwhile, we find that T_2 is bounded above by the number of integral solutions of the equation

$$x_1^3 - x_2^3 = n_1 - n_2, \tag{9.9}$$

with $1 \leqslant x_1, x_2 \leqslant P$ and $n_1, n_2 \in \mathcal{Z}(N)$. The number of such solutions with $n_1 = n_2$ and $x_1 = x_2$ is plainly PZ. When $n_1 \neq n_2$, it is possible that the integer $n_1 - n_2$ is represented uniquely as the difference of two cubes of natural numbers. Plainly, the number of such solutions of (9.9) is $O(Z^2)$. Let S denote the number of positive integral solutions of the equation

$$y_1^3 - y_2^3 = y_3^3 - y_4^3$$

with $1 \leqslant y_i \leqslant P$ $(1 \leqslant i \leqslant 4)$ and $y_1 \notin \{y_2, y_3\}$. Then it follows from work of Heath-Brown [12] that $S = O(P^{4/3+\varepsilon})$. We therefore deduce that there are at most $O(P^{4/3+\varepsilon})$ integers m that have two or more representations in the shape $m = x_1^3 - x_2^3$, with $1 \leqslant x_1, x_2 \leqslant P$. Since each such integer m has at most Z representations in the shape $m = n_1 - n_2$, with $n_1, n_2 \in \mathcal{Z}(N)$, we conclude that the contribution to T_2 arising from this class of solutions of (9.9) is $O(ZP^{4/3+\varepsilon})$. Combining the above estimates, we arrive at the upper bound

$$T_2 \ll Z^2 + ZP^{4/3+\varepsilon}. \tag{9.10}$$

On substituting (9.8) and (9.10) into (9.6), we find that

$$N^{4/3}\psi(N)^{-1}Z \ll (P^8(\log P)^{\varepsilon-2})^{1/2}(Z^2 + ZP^{4/3+\varepsilon})^{1/2},$$

whence

$$Z \ll Z\psi(N)(\log N)^{\varepsilon-1} + Z^{1/2}N^{2/9+\varepsilon}\psi(N).$$

Thus, whenever $\psi(N) = O((\log N)^{1-\delta})$, it follows that

$$Z \ll N^{4/9+2\varepsilon}\psi(N)^2 \ll N^{4/9+3\varepsilon}.$$

On summing over dyadic intervals, we conclude that $\widetilde{E}_7(N;\psi) \ll N^{4/9+\varepsilon}$, and this completes the proof of Theorem 1.3.

We emphasise the crucial role played by the estimate $S = O(P^{4/3+\varepsilon})$ in the above argument. If one were able to replace this consequence of Heath-Brown's work with the new estimate $S = O_\varepsilon(P^{\xi+\varepsilon})$, then one obtains $\widetilde{E}_7(N;\psi) \ll N^{\xi/3+\varepsilon}$. The latter estimate supersedes the classical estimate only for $\xi < 3/2$, while the conjectured permissible exponent $\xi = 1$ yields $\widetilde{E}_7(N;\psi) \ll N^{1/3+\varepsilon}$.

10. Conditional conclusions for sums of six cubes. We conclude this paper with an account of the proof of Theorem 1.4, and this entails some further auxiliary estimates. The prerequisites for Theorem 1.4 may be cheaply disposed of by exploiting the work of §§2 and 3 of Brüdern, Kawada and Wooley [4]. We begin with some notation. Let N be a large positive number, and write $P = \frac{1}{2}N^{1/3}$. We take η to be a sufficiently small positive number depending at most on ε, and we consider a real number R with $P^{\eta/2} < R \leqslant P^\eta$. Denote by $\mathcal{A}(X,R)$ the set of R-smooth numbers not exceeding X, that is

$$\mathcal{A}(X,R) = \{n \in [1,X] \cap \mathbb{Z} : p \text{ prime}, p|n \Rightarrow p \leqslant R\}.$$

Also, put $Q = P^{6/7}$ and $Y = P^{1/7}$. We then define the generating functions

$$f(\alpha) = \sum_{P<x\leqslant 2P} e(\alpha x^3), \quad f_p(\alpha) = \sum_{\substack{P<x\leqslant 2P\\(x,p)=1}} e(\alpha x^3),$$

$$g(\alpha) = \sum_{Q<y\leqslant 2Q} e(\alpha y^3), \quad h(\alpha) = \sum_{z\in\mathcal{A}(Q,R)} e(\alpha z^3).$$

Following Brüdern, Kawada and Wooley [4], we define

$$\mathcal{F}(\alpha) = \sum_{\substack{Y<p\leqslant 2Y\\ p\equiv 2 \pmod 3}} f_p(\alpha)g(\alpha p^3)h(\alpha p^3)^2,$$

and then put

$$\mathcal{S}(\alpha) = f(\alpha)^2 \mathcal{F}(\alpha). \tag{10.1}$$

We require two different Hardy-Littlewood dissections, and in this context we write $L = (\log P)^{1/100}$, and define $\mathfrak{N}$ to be the union of the intervals

$$\mathfrak{N}(q, a) = \{\alpha \in [0, 1) : |q\alpha - a| \leqslant LP^{-3}\},$$

with $0 \leqslant a \leqslant q \leqslant L$ and $(a, q) = 1$. Also, we write $\mathfrak{n} = [0, 1) \setminus \mathfrak{N}$. Before defining our second dissection, we pause to record a major arc estimate.

Lemma 10.1. *Uniformly for $N/2 < m \leqslant N$, one has the estimate*

$$\int_{\mathfrak{N}} \mathcal{S}(\alpha) e(-\alpha m) d\alpha \gg_\eta YQ^3 (\log P)^{-1}.$$

Proof. The desired lower bound follows easily via the argument of the proof of Lemma 2.1 of [4]. In the present situation, the generating function $\mathcal{S}(\alpha)$ contains two copies of the complete exponential sum $f(\alpha)$, as opposed to only one in the analogous situation in the latter paper (although therein our second copy of $f(\alpha)$ is replaced by a smooth Weyl sum). However, this modification in fact facilitates the details of the analysis at hand.

Next define the set of major arcs $\mathfrak{M}$ to be the union of the intervals

$$\mathfrak{M}(q, a) = \{\alpha \in [0, 1) : |q\alpha - a| \leqslant P^{-9/4}\},$$

with $0 \leqslant a \leqslant q \leqslant P^{3/4}$ and $(a, q) = 1$. Define also $\mathfrak{m} = [0, 1) \setminus \mathfrak{M}$.

Lemma 10.2. *One has*

$$\int_{\mathfrak{m}} |\mathcal{F}(\alpha)|^2 d\alpha \ll Y^2 Q^6 P^{-19/14}.$$

Moreover, there is a function $\mathcal{F}_1 : [0, 1) \to \mathbb{C}$ with the property that

$$\int_{\mathfrak{M}} |\mathcal{F}(\alpha) - \mathcal{F}_1(\alpha)|^2 d\alpha \ll Y^2 Q^6 P^{-19/14},$$

and satisfying the condition that

$$\int_{\mathfrak{n} \cap \mathfrak{M}} |\mathcal{F}_1(\alpha) f(\alpha)^2| d\alpha \ll Q^3 Y L^{-1/4} (\log Y)^{-1}.$$

Proof. The first two conclusions are immediate from Lemma 3.2 of [4]. The final estimate, meanwhile, is essentially the inequality (3.13) of the latter paper.

We are now equipped to establish Theorem 1.4. Let $\mathcal{Z}(N)$ denote the set of integers n with $N/2 < n \leqslant N$ for which the equation

$$x_1^3 + \cdots + x_6^3 = n$$

has no solution in positive integers x_i $(1 \leqslant i \leqslant 6)$. We define the exponential sum $K(\alpha)$ by

$$K(\alpha) = \sum_{n \in \mathcal{Z}(N)} e(n\alpha),$$

and we abbreviate $\text{card}(\mathcal{Z}(N))$ to Z. Then by orthogonality, it is apparent that

$$\int_0^1 \mathcal{S}(\alpha) K(-\alpha) d\alpha = \sum_{n \in \mathcal{Z}(N)} \int_0^1 \mathcal{S}(\alpha) e(-n\alpha) d\alpha = 0.$$

Then Lemma 10.1 leads to the relation

$$\begin{aligned}\int_{\mathfrak{N}} \mathcal{S}(\alpha)K(-\alpha)d\alpha &= \sum_{n\in\mathcal{Z}(N)} \int_{\mathfrak{N}} \mathcal{S}(\alpha)e(-n\alpha)d\alpha \\ &\gg ZYQ^3(\log P)^{-1},\end{aligned}$$

whence we deduce that

$$\Big|\int_{\mathfrak{n}} \mathcal{S}(\alpha)K(-\alpha)d\alpha\Big| \gg ZYQ^3(\log P)^{-1}. \tag{10.2}$$

Next, on recalling (10.1) we find that

$$\Big|\int_{\mathfrak{n}} \mathcal{S}(\alpha)K(-\alpha)d\alpha\Big| \leqslant U_1+U_2+U_3, \tag{10.3}$$

where

$$U_1=\int_{\mathfrak{n}\cap\mathfrak{M}} |\mathcal{F}_1(\alpha)f(\alpha)^2K(\alpha)|d\alpha, \tag{10.4}$$

$$U_2=\int_{\mathfrak{M}} |(\mathcal{F}(\alpha)-\mathcal{F}_1(\alpha))f(\alpha)^2K(\alpha)|d\alpha, \tag{10.5}$$

$$U_3=\int_{\mathfrak{m}} |\mathcal{F}(\alpha)f(\alpha)^2K(\alpha)|d\alpha. \tag{10.6}$$

But the final conclusion of Lemma 10.2 reveals that

$$U_1\leqslant K(0)\int_{\mathfrak{n}\cap\mathfrak{M}} |\mathcal{F}_1(\alpha)f(\alpha)^2|d\alpha \ll ZQ^3YL^{-1/4}(\log Y)^{-1},$$

whence it follows from (10.2) and (10.3) that

$$ZYQ^3(\log P)^{-1}\ll U_2+U_3. \tag{10.7}$$

On writing

$$U_4=\int_0^1 |f(\alpha)^4K(\alpha)^2|d\alpha,$$

we next deduce from Schwarz's inequality together with (10.5) and (10.6) that

$$U_2+U_3\leqslant U_4^{1/2}\Big(\int_{\mathfrak{M}} |\mathcal{F}(\alpha)-\mathcal{F}_1(\alpha)|^2d\alpha+\int_{\mathfrak{m}} |\mathcal{F}(\alpha)|^2d\alpha\Big)^{1/2}.$$

Then in view of the first two conclusions of Lemma 10.2, we may conclude from (10.7) that

$$ZYQ^3(\log P)^{-1}\ll U_4^{1/2}(Y^2Q^6P^{-19/14})^{1/2},$$

whence

$$Z\ll U_4^{1/2}P^{-19/28}\log P. \tag{10.8}$$

The mean value U_4 can be estimated with the aid of Hypothesis $\mathcal{R}(A)$, and this we discuss in a general context via the following lemma.

Lemma 10.3. *Let A be a positive real number, and suppose that the hypothesis $\mathcal{R}(A)$ holds. Then for each $\varepsilon>0$, one has*

$$\int_0^1 |f(\alpha)^4K(\alpha)^2|d\alpha \ll ZP^2+Z^2P^{A+\varepsilon}. \tag{10.9}$$

Proof. The mean value on the left hand side of (10.9) counts the number of solutions of the equation

$$x_1^3+x_2^3-x_3^3-x_4^3=n_1-n_2,$$

with $P < x_i \leqslant 2P$ $(1 \leqslant i \leqslant 4)$ and $n_1, n_2 \in \mathcal{Z}(N)$. Given any one of the $O(Z^2)$ possible choices for n_1, n_2 with $n_1 \neq n_2$, it follows from hypothesis $\mathcal{R}(A)$ that the number of permissible choices for $\mathbf{x}$ is $O(P^{A+\varepsilon})$, and thus the contribution arising from this class of solutions is $O(Z^2 P^{A+\varepsilon})$. When $n_1 = n_2$, on the other hand, the variables $\mathbf{x}$ satisfy the equation $x_1^3 + x_2^3 = x_3^3 + x_4^3$. Here, one finds from Hooley [14], for example, that the number of solutions with $1 \leqslant x_i \leqslant 2P$ $(1 \leqslant i \leqslant 4)$ is $O(P^2)$. Thus we conclude that the number of solutions of this type is $O(ZP^2)$. The estimate (10.9) is now immediate on combining the above upper bounds.

On substituting the conclusion of Lemma 10.3, with $A = \xi < 19/14$, into the relation (10.8), we find that

$$\begin{aligned} Z &\ll (ZP^2 + Z^2 P^{19/14} (\log P)^{-4})^{1/2} P^{-19/28} \log P \\ &\ll Z^{1/2} P^{9/28} \log P + Z(\log P)^{-1}. \end{aligned}$$

We therefore conclude that

$$Z \ll P^{9/14} (\log P)^2 \asymp N^{3/14} (\log N)^2,$$

and the conclusion of Theorem 1.4 follows on summing over dyadic intervals.

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